

Symplectic methods based on Padé approximation for some stochastic Hamiltonian systems

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Abstract

In this article, we introduce a kind of numerical schemes, based on Padé approximation, for two stochastic Hamiltonian systems which are treated separately. For the linear stochastic Hamiltonian systems, it is shown that the applied Padé approximations $P_{(k,k)}$ give numerical solutions that inherit the symplecticity and the proposed numerical schemes based on $P_{(r,s)}$ are of mean-square order $\frac{r+s}{2}$ under appropriate conditions. In case of the special stochastic Hamiltonian systems with additive noises, the numerical method using two kinds of Padé approximation $P_{(\hat{r},\hat{s})}$ and $P_{(\check{r},\check{s})}$ has mean-square order $\check{r} + \check{s} + 1$ when $\hat{r} + \hat{s} = \check{r} + \check{s} + 2$. Moreover, the numerical solution is symplectic if $\hat{r} = \hat{s}$.

Keywords: Padé approximation; linear stochastic Hamiltonian systems; symplecticity; mean-square order;

1 Introduction

It is well known that the flow $\varphi(t)$ of the deterministic Hamiltonian system

$$\begin{aligned} dp &= -\frac{\partial H(p, q)}{\partial q} dt, & p(0) &= p_0, \\ dq &= \frac{\partial H(p, q)}{\partial p} dt, & q(0) &= q_0, \end{aligned} \tag{1}$$

is symplectic, i.e.

$$dp(t) \wedge dq(t) = dp_0 \wedge dq_0, \forall t \geq 0,$$

which also can be depicted as follows

$$\left(\frac{\partial \varphi(t)}{\partial y_0} \right)^\top J \left(\frac{\partial \varphi(t)}{\partial y_0} \right) = J, \quad y_0 = (p_0, q_0)^\top.$$

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In recent years, many numerical analysts have been devoting to constructing symplectic numerical methods for such systems, that is, numerical approximations (p_n, q_n) that preserve the symplecticity of the underlying Hamiltonian systems, characterised by

$$dp_{n+1} \wedge dq_{n+1} = dp_n \wedge dq_n, \quad \forall n \in \mathbb{Z}, n \geq 0.$$

For example, the numerical methods based on the Páde approximation are effective symplectic solvers of linear Hamiltonian systems (LHS) [1], i.e. the Hamiltonian system (1) where Hamiltonian function $H(p, q)$ is of the quadratic form

$$H(p, q) = \frac{1}{2}(p^\top, q^\top)C \begin{pmatrix} p \\ q \end{pmatrix}, \quad C^\top = C.$$

For the LHS, symplectic numerical schemes based on the Padé approximation $P_{(k,k)}(C)$ take the form

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = P_{(k,k)}(C) \begin{bmatrix} p_n \\ q_n \end{bmatrix} = [D_{(k,k)}(C)]^{-1} N_{(k,k)}(C) \begin{bmatrix} p_n \\ q_n \end{bmatrix}, \quad (2)$$

with the matrix polynomials

$$\begin{aligned} N_{(k,k)}(C) &= \sum_{i=0}^k \frac{(2k-i)!k!}{(2k)!i!(k-i)!} (hJ^{-1}C)^i, \\ D_{(k,k)}(C) &= \sum_{i=0}^k \frac{(2k-i)!k!}{(2k)!i!(k-i)!} (-hJ^{-1}C)^i, \end{aligned} \quad (3)$$

and a standard antisymmetric matrix $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ for $k = 1, 2, \dots$. It is proved that the difference scheme (2) is not only symplectic, but also of $2k$ -th order of accuracy ([1]).

Stochastic Hamiltonian systems (SHSs) constitute a rather important class of stochastic systems having the property of symplecticity ([5, 6, 7]). The problem of constructing special numerical methods preserving the symplectic structure of SHSs is of great current interest ([3, 5, 6, 7]). We consider the linear stochastic Hamiltonian systems (4) of dimension $d = 2n$ in the sense of Stratonovich

$$\begin{aligned} dP &= -\frac{\partial H_0(P, Q)}{\partial Q} dt - \sum_{i=1}^m \frac{\partial H_i(P, Q)}{\partial Q} \circ dW^i(t), \quad P(t_0) = p, \\ dQ &= \frac{\partial H_0(P, Q)}{\partial P} dt + \sum_{i=1}^m \frac{\partial H_i(P, Q)}{\partial P} \circ dW^i(t), \quad Q(t_0) = q, \end{aligned} \quad (4)$$

and stochastic Hamiltonian systems with additive noises in the sense of Itô (5)

$$\begin{aligned} d\tilde{P} &= -\frac{\partial \tilde{H}_0(\tilde{P}, \tilde{Q})}{\partial \tilde{Q}} dt - \sum_{i=1}^m \frac{\partial \tilde{H}_i(\tilde{P}, \tilde{Q})}{\partial \tilde{Q}} dW^i(t), \quad \tilde{P}(t_0) = \tilde{p}, \\ d\tilde{Q} &= \frac{\partial \tilde{H}_0(\tilde{P}, \tilde{Q})}{\partial \tilde{P}} dt + \sum_{i=1}^m \frac{\partial \tilde{H}_i(\tilde{P}, \tilde{Q})}{\partial \tilde{P}} dW^i(t), \quad \tilde{Q}(t_0) = \tilde{q}, \end{aligned} \quad (5)$$

where $P, Q, \tilde{P}, \tilde{Q}, p, q, \tilde{p}$ and \tilde{q} are n -dimensional column-vectors with the components $P_l, Q_l, \tilde{P}_l, \tilde{Q}_l, p, q, \tilde{p}_l, \tilde{q}_l, l = 1, \dots, n$, and $(W^1(t), \dots, W^m(t))$ is m - n -dimensional standard Wiener process. In the stochastic differential equations (4) and (5), $H_i(P, Q), i = 0, \dots, m, \tilde{H}_0(\tilde{P}, \tilde{Q})$ are of quadratic forms, i.e.

$$H_i(P, Q) = \frac{1}{2}(P^\top, Q^\top)C^i \begin{pmatrix} P \\ Q \end{pmatrix}, \quad \tilde{H}_0(\tilde{P}, \tilde{Q}) = \frac{1}{2}(\tilde{P}^\top, \tilde{Q}^\top)\tilde{C}^0 \begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix},$$

where $C^i, i = 0, \dots, m$ and \tilde{C}^0 are $2n \times 2n$ symmetric matrices.

$$\tilde{H}_i = \langle \tilde{C}_1^i, \tilde{P} \rangle - \langle \tilde{C}_2^i, \tilde{Q} \rangle, \quad i = 1, \dots, m$$

where $\tilde{C}_1^i, \tilde{C}_2^i, i = 1, \dots, m$ are n -dimensional constant column-vectors. In addition, we suppose that the coefficients of the LSHS are sufficiently smooth functions defined for $(t; p, q) \in [t_0, t_0 + T] \times R^{2n}$ which guarantee the existence and the uniqueness of the solution in the interval $[t_0, t_0 + T]$ (see [4, 8]).

Let $X(t; t_0, p, q) = (P(t; t_0, p, q)^\top, Q(t; t_0, p, q)^\top)^\top, t_0 \leq t \leq t_0 + T$ and $Z(t; t_0, \tilde{p}, \tilde{q}) = (\tilde{P}(t; t_0, \tilde{p}, \tilde{q})^\top, \tilde{Q}(t; t_0, \tilde{p}, \tilde{q})^\top)^\top, t_0 \leq t \leq t_0 + T$ be the solutions of the linear stochastic Hamiltonian system (4) and the linear stochastic Hamiltonian system with additive noises (5) respectively. A more detailed notation is $X(t; t_0, p, q, \omega)$ and $Z(t; t_0, \tilde{p}, \tilde{q}, \omega)$, where ω is an elementary event. It is known that $X(t; t_0, p, q)$ and $Z(t; t_0, \tilde{p}, \tilde{q})$ are phase flows (diffeomorphism) for almost every ω , properties of which can be seen in e.g., [9]. Moreover, we denote by $X_k = (P_k^\top, Q_k^\top)^\top, k = 0, \dots, N$ the numerical method for (4) and by $Z_k = (\tilde{P}_k^\top, \tilde{Q}_k^\top)^\top, k = 0, \dots, N$ the numerical method for (5), respectively, and $h = t_{k+1} - t_k, t_N = t_0 + T$.

As the class of the symplectic difference schemes (2) for the deterministic Hamiltonian systems (1) have been proposed, it is meaningful to investigate, whether we could extend this approach to stochastic context and construct the symplectic numerical methods for stochastic Hamiltonian systems (4) and (5) by using the Padé approximation. An outline of this paper is as follows: Section 2 is devoted to numerical methods $X^{(r,s)}(t+h; t, p, q), m = 1, 2, 3, 4$ via using the Padé approximation for linear stochastic Hamiltonian systems and a review of well known facts concerning some properties of the infinitesimal symplectic matrices. In Section 3, some numerical schemes $Z(t+h; t, \tilde{p}, \tilde{q})$ based on the Padé. approximation $P_{(\tilde{r}, \tilde{s})}$ and $P_{(\tilde{r}, \tilde{s})}$ are proposed for the special stochastic Hamiltonian systems with additive noises. In Section 4, we investigate the mean-square convergence of the proposed numerical approximations and prove that some of them preserve symplectic structure under appropriate conditions. Section 5 gives numerical tests. At last, Section 6 is a brief conclusion

2 Symplectic numerical methods for linear Stochastic Hamiltonian systems

2.1 Linear Stochastic Hamiltonian systems(LSHS)

Using $\nabla H_i(X) = C^i X = JA^i X$, $i = 0, 1, \dots, m$, the canonical system (4) becomes

$$dX(t) = A^0 X dt + \sum_{i=1}^m A^i X \circ dW^i(t), \quad X(t_0) = (p, q)^\top, \quad (6)$$

the unique solution which is as follows:

$$X(t) = \exp \left[(t - t_0)A^0 + \sum_{i=1}^m (W^i(t) - W^i(t_0))A^i \right] X(t_0), \quad (7)$$

with $t_0 \leq t \leq t_0 + T$. Denoting

$$A^i = \begin{bmatrix} A_1^i & A_2^i \\ A_3^i & A_4^i \end{bmatrix}, i = 0, 1, \dots, m,$$

where A_j^i , $i = 0, \dots, m$, $j = 1, 2, 3, 4$ are $n \times n$ constant matrices, and substituting the symmetric matrices A^i into the right side of (6), we obtain

$$\begin{aligned} dP &= (A_1^0 P + A_2^0 Q)dt + \sum_{i=1}^m (A_1^i P + A_2^i Q) \circ dW^i(t), \quad P(t_0) = p, \\ dQ &= (A_3^0 P + A_4^0 Q)dt + \sum_{i=1}^m (A_3^i P + A_4^i Q) \circ dW^i(t), \quad Q(t_0) = q. \end{aligned} \quad (8)$$

Since JA^i , $i = 0, 1, \dots, m$ are symmetric matrices, it is easy to check that $A_1^i = -A_4^{i\top}$, $A_2^i = A_2^{i\top}$, $A_3^i = A_3^{i\top}$, $i = 0, \dots, m$.

According to the fundamental theorem of Hamiltonian systems, the solution of a Hamiltonian system is a one-parameter symplectic group G_t whose elements are called symplectic matrices, denoted by $Sp(2n)$. Therefore, the symplectic geometry serves as the mathematical foundation of Hamiltonian mechanics. Before stating the numerical methods for LSHS, let us introduce some properties of infinitesimal symplectic matrices, which will be used in the proof of our main results Theorem 4.3 and Theorem 4.4.

Definition 2.1. ([1]) A matrix B of order $2n$ is called symplectic, i.e. $B \in Sp(2n)$, if

$$B^\top JB = J.$$

where B^\top is the transpose of B . All symplectic matrices form a symplectic group $Sp(2n)$.

Definition 2.2. ([1]) A matrix B of order $2n$ is called infinitesimal symplectic, if

$$JB + B^\top J = O.$$

All infinitesimal symplectic matrices form a Lie algebra with commutation operation $[A, B] = AB - BA$, denoted as $sp(2n)$. $sp(2n)$ is the Lie algebra of the Lie group $Sp(2n)$.

Remark 2.1. The matrices A^i , $i = 0, 1, \dots, m$, in (6) are infinitesimal symplectic matrices.

Theorem 2.1. ([1]) If $f(x)$ is an even polynomial, and $B \in sp(2n)$, then

$$f(B^\top)J = Jf(B).$$

Theorem 2.2. ([1]) If $g(x)$ is an odd polynomial, and $B \in sp(2n)$, then $g(B) \in sp(2n)$, i.e.,

$$g(B^\top)J + Jg(B) = O.$$

Theorem 2.3. ([1]) Matrices $S = M^{-1}N \in Sp(2n)$, iff

$$MJM^\top = NJN^\top. \quad (9)$$

2.2 Constructing numerical methods based on Padé approximation

As is well known, the exponential function $\exp(M)$ for $n \times n$ dimensional matrix M has the Taylor's expansion

$$\exp(M) = I + \sum_{i=1}^{+\infty} \frac{M^i}{i!}, \quad (10)$$

and can be approximated by Padé approximation as follows

$$\exp(M) \sim P_{(r,s)}(M) = D_{(r,s)}^{-1}(M)N_{(r,s)}(M), \quad (11)$$

with

$$N_{(r,s)}(M) = I + \sum_{i=1}^r \frac{(r+s-i)!r!}{(r+s)!i!(r-i)!} M^i = I + \sum_{i=1}^r a_i M^i,$$

,

$$D_{(r,s)}(M) = I + \sum_{i=1}^s \frac{(r+s-i)!s!}{(r+s)!i!(s-i)!} (-M)^i = I + \sum_{i=1}^s b_i (-M)^i.$$

In addition, (11) makes the following equation holds ([1])

$$\exp(M) - P_{(r,s)}(M) = c_{r+s+1}M^{r+s+1} + \sum_{i=r+s+2}^{+\infty} c_i M^i = O(M^{r+s+1}). \quad (12)$$

where c_i , $i \geq r + s + 1$ are constants.

Based on the Padé approximation $P_{(r,s)}[(t_{n+1}-t_n)A^0 + \sum_{i=1}^m (W^i(t_{n+1})-W^i(t_n))A^i]$, we construct the following numerical methods for linear stochastic Hamiltonian systems (6)

$$\hat{X}_{n+1}^{(r,s)} = \left[I + \sum_{j=1}^s b_j (-B)^j \right]^{-1} \left[I + \sum_{j=1}^r a_j (B)^j \right] \hat{X}_n^{(r,s)}, \quad (13)$$

where $B = hA^0 + \sum_{i=1}^m \Delta W_n^i A^i$ and $\Delta W_n^i = W^i(t_{n+1}) - W^i(t_n)$ are the increments of Wiener processes which can be substituted by $\sqrt{h}\xi^i$ and $\xi^i \sim N(0, 1)$, $i = 1, \dots, m$ are independent random variables.

Similar to the discussion in ([7]), we truncate ξ^i to another random variable ζ_h^i which is bounded. In detail,

$$\zeta_h^i = \begin{cases} \xi^i, & \text{if } |\xi^i| \leq A_h, \\ A_h, & \text{if } \xi^i > A_h, \\ -A_h, & \text{if } \xi^i < -A_h, \end{cases}$$

where $A_h = \sqrt{2\ell|\ln h|}$, $\ell \geq 1$. For the sake of simplicity, we denote the matrix polynomials $hA^0 + \sum_{i=1}^m \sqrt{h}\zeta_h^i A^i$ by \bar{B} , and the following numerical scheme:

$$X_{n+1}^{(r,s)} = \left[I + \sum_{j=1}^s b_j (-\bar{B})^j \right]^{-1} \left[I + \sum_{j=1}^r a_j \bar{B}^j \right] X_n^{(r,s)}. \quad (14)$$

with $A_h^i = \sqrt{2\ell|\ln h|}$, $\ell \geq 1$. It is interesting to observe that if $r = s = 1$, we reattain the Euler centered scheme

$$X_{n+1}^{(1,1)} = X_n^{(1,1)} + \frac{1}{2}\bar{B}(X_n^{(1,1)} + X_{n+1}^{(1,1)}). \quad (15)$$

If both r and s take the value 2, we obtain the scheme

$$X_{n+1}^{(2,2)} = X_n^{(2,2)} + \frac{1}{2}\bar{B}(X_n^{(2,2)} + X_{n+1}^{(2,2)}) + \frac{1}{12}\bar{B}^2(X_n^{(2,2)} - X_{n+1}^{(2,2)}). \quad (16)$$

Similarly, if $r = s$ equal 3 and 4, the following two kinds of numerical approximations arise, respectively,

$$X_{n+1}^{(3,3)} = X_n^{(3,3)} + \left(\frac{1}{2}\bar{B} + \frac{1}{120}\bar{B}^3\right)(X_n^{(3,3)} + X_{n+1}^{(3,3)}) + \frac{1}{10}\bar{B}^2(X_n^{(3,3)} - X_{n+1}^{(3,3)}). \quad (17)$$

$$X_{n+1}^{(4,4)} = X_n^{(4,4)} + \left(\frac{1}{2}\bar{B} + \frac{1}{84}\bar{B}^3\right)(X_n^{(4,4)} + X_{n+1}^{(4,4)}) + \left(\frac{1}{24}\bar{B}^2 + \frac{1}{1680}\bar{B}^4\right)(X_n^{(4,4)} - X_{n+1}^{(4,4)}). \quad (18)$$

In Section 4, the properties of the methods (15) \sim (18) will be analyzed.

3 Symplectic numerical methods for stochastic Hamiltonian systems with additive noises

We now turn our attention to stochastic Hamiltonian systems with additive noises (5) mentioned in Section 1,

$$\begin{aligned} d\tilde{P} &= -\frac{\partial \tilde{H}_0(\tilde{P}, \tilde{Q})}{\partial \tilde{Q}} dt - \sum_{i=1}^m \frac{\partial \tilde{H}_i(\tilde{P}, \tilde{Q})}{\partial \tilde{Q}} dW^i(t), \quad \tilde{P}(t_0) = \tilde{p}, \\ d\tilde{Q} &= \frac{\partial \tilde{H}_0(\tilde{P}, \tilde{Q})}{\partial \tilde{P}} dt + \sum_{i=1}^m \frac{\partial \tilde{H}_i(\tilde{P}, \tilde{Q})}{\partial \tilde{P}} dW^i(t), \quad \tilde{Q}(t_0) = \tilde{q}, \end{aligned}$$

where

$$\tilde{H}_0(\tilde{P}, \tilde{Q}) = \frac{1}{2}(\tilde{P}^\top, \tilde{Q}^\top) \tilde{C}^0 \begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix}, \quad \tilde{C}^0 = \tilde{C}^{0\top},$$

and

$$\tilde{H}_i = \langle \tilde{C}_1^i, \tilde{P} \rangle - \langle \tilde{C}_2^i, \tilde{Q} \rangle, \quad i = 1, \dots, m.$$

By $\nabla \tilde{H}_0(Z) = \tilde{C}^0 Z$ and $\nabla \tilde{H}_i(Z) = R_i = (\tilde{C}_1^{i\top}, -\tilde{C}_2^{i\top})^\top$, the stochastic Hamiltonian system with additive noises (5) becomes

$$dZ(t) = J^{-1} \tilde{C}^0 Z dt + \sum_{i=1}^m J^{-1} R_i dW^i(t), \quad Z(t_0) = (\tilde{p}, \tilde{q})^\top, \quad (19)$$

the exact solution of which is as follows:

$$Z(t) = e^{(t-t_0)J^{-1}\tilde{C}^0} Z(t_0) + \sum_{i=1}^m \int_{t_0}^t e^{(t-\theta)J^{-1}\tilde{C}^0} J^{-1} R_i dW^i(\theta), \quad (20)$$

with $t_0 \leq t \leq t_0 + T$.

An approach for the construction of the numerical schemes for stochastic differential equations (19) is to replace the matrix exponential by the Padé approximation. Approximating the matrix exponential $e^{(t-t_0)J^{-1}\tilde{C}^0}$ and $e^{(t-s)J^{-1}\tilde{C}^0}$ by the Padé approximations $P_{(\hat{r}, \hat{s})}[(t_{n+1} - t_n)J^{-1}\tilde{C}^0]$ and $P_{(\check{r}, \check{s})}[(t_{n+1} - s)J^{-1}\tilde{C}^0]$ respectively, we obtain the numerical methods as follows

$$\begin{aligned} Z_{n+1} &= \left[I + \sum_{j=1}^{\hat{s}} b_j (-B_1)^j \right]^{-1} \left[I + \sum_{j=1}^{\hat{r}} a_j (B_1)^j \right] Z_n \\ &\quad + \sum_{i=1}^m \int_{t_n}^{t_{n+1}} \left[I + \sum_{j=1}^{\check{s}} b_j (-B_2)^j \right]^{-1} \left[I + \sum_{j=1}^{\check{r}} a_j (B_2)^j \right] J^{-1} R_i dW^i(\theta), \end{aligned} \quad (21)$$

where $B_1 = hJ^{-1}\tilde{C}^0$ and $B_2 = (t_{n+1} - \theta)J^{-1}\tilde{C}^0$ with $\hat{r} + \hat{s} = \check{r} + \check{s} + 2$ and $\check{r}, \check{s} \geq 1$. In particular, if $\hat{r} = \hat{s}$, the one-step approximation is symplectic which will be proved in Section 4.

Approximating Itô integral in (20) by the left-rectangle formula and the Padé approximation $P_{(1,1)}(hJ^{-1}\bar{C}^0)$, we obtain the explicit numerical scheme

$$\begin{aligned} Z_{n+1} = & \left[I + \sum_{j=1}^{\hat{s}} b_j (-B_1)^j \right]^{-1} \left[I + \sum_{j=1}^{\hat{r}} a_j (B_1)^j \right] Z_n \\ & + \sum_{i=1}^m \left[I - \frac{1}{2} B_1 \right]^{-1} \left[I + \frac{1}{2} B_1 \right] J^{-1} R_i \Delta W_n^i, \end{aligned} \quad (22)$$

with every $\hat{s}, \hat{r} \geq 1$. Similar to the special case of the numerical scheme (21), the one-step approximation (22) preserves symplecticity if $\hat{r} = \hat{s}$. Moreover, we can prove that the numerical method (22) is of mean-square order 1.

4 Properties of the numerical methods based on the Padé approximation

In this section, we prove the mean-square convergence order of the proposed schemes (15) ~ (18) and (21) based on the Padé approximation, as well as their symplecticity. For the former purpose, we need the fundamental convergence theorem in stochastic context given by G. N. Milstein and M.V. Tretyakov([6, 7, 5]).

Proposition 4.1. (see [5, 6, 7]) Suppose the one-step approximation $\bar{X}(t+h; t, x)$ has the order of accuracy p_1 for the mathematical expectation of the deviation and order of accuracy p_2 for the mean-square deviation; more precisely, for arbitrary $t_0 \leq t \leq t_0 + T - h$, $x \in R^n$ the following inequalities hold:

$$\begin{aligned} |\mathbf{E}(X(t+h; t, x) - \bar{X}(t+h; t, x))| & \leq K(1 + |x|^2)^{\frac{1}{2}} h^{p_1}, \\ \left[\mathbf{E}|X(t+h; t, x) - \bar{X}(t+h; t, x)|^2 \right]^{\frac{1}{2}} & \leq K(1 + |x|^2)^{\frac{1}{2}} h^{p_2}, \end{aligned} \quad (23)$$

Also, let

$$p_2 \geq \frac{1}{2}, p_1 \geq p_2 + \frac{1}{2},$$

Then for any N and $k = 0, \dots, N$ the following inequality holds:

$$\left[\mathbf{E}|X(t_k; t_0, X_0) - \bar{X}(t_k; t_0, X_0)|^2 \right]^{\frac{1}{2}} \leq K(1 + |X_0|^2)^{\frac{1}{2}} h^{p_2 - \frac{1}{2}}, \quad (24)$$

i.e. the order of accuracy of the method constructed using the one-step approximation $\bar{X}(t+h; t, x)$ is $p = p_2 - \frac{1}{2}$.

We note that all the constants K mentioned above, as well as the ones that will appear in the sequels, depend on the system and the approximation only and do not depend on X_0 and N .

Proposition 4.2. (see [5, 6, 7]) Let the one-step approximation $\bar{X}(t+h; t, x)$ satisfy the condition of Theorem 4.1. Suppose that $\tilde{X}(t+h; t, x)$ is such that

$$\begin{aligned} |\mathbf{E}(\bar{X}(t+h; t, x) - \tilde{X}(t+h; t, x))| &= O(h^{p_1}), \\ \left[\mathbf{E}|\bar{X}(t+h; t, x) - \tilde{X}(t+h; t, x)|^2 \right]^{\frac{1}{2}} &= O(h^{p_2}), \end{aligned} \quad (25)$$

with the same h^{p_1} and h^{p_2} . Then the method based on the one-step approximation $\tilde{X}(t+h; t, x)$ has the same mean-square order of accuracy as the method based on $\bar{X}(t+h; t, x)$, i.e., its order is equal to $p = p_2 - \frac{1}{2}$.

Our result regarding the mean-square convergence order of the schemes (15) \sim (18) and (21) is as follows.

Theorem 4.1. The numerical methods $X_n^{(r,s)}$ e.g. (15) \sim (18) based on the Padé approximation $P_{(r,s)}$ with $A_h^i = \sqrt{2\ell \ln h}$, $\ell \geq r+s$ is of mean-square order $\frac{r+s}{2}$.

Proof. According to an analog of the Taylor expansion of the solution $X(t+h; t, x)$ in (7), we obtain a one-step approximation $\hat{X}(t+h; t, x)$ as follows

$$\begin{aligned} \hat{X} &= x + \sum_{j=1}^{r+s} \frac{1}{j!} (hA^0 + \sum_{i=1}^m \sqrt{h}\zeta_h^i A^i)^j x \\ &= x + \sum_{j=1}^{r+s} \frac{1}{j!} \bar{B}^j x, \end{aligned} \quad (26)$$

which has the $\frac{r+s}{2}$ -th mean-square order of convergence according to the Proposition 4.1 by verifying the $|\mathbf{E}(X - \hat{X})|$ and $\mathbf{E}|(X - \hat{X})|^2$. First,

$$\begin{aligned} &|\mathbf{E}(X - \hat{X})| \\ &\leq K \left| \sum_{j=1}^{r+s} \frac{1}{j!} [\mathbf{E}(hA^0 + \sum_{i=1}^m \sqrt{h}\xi^i A^i)^j - \mathbf{E}(hA^0 + \sum_{i=1}^m \sqrt{h}\zeta_h^i A^i)^j] \right| \\ &\quad + K \left| \sum_{j=r+s+1}^{+\infty} \frac{1}{j!} \mathbf{E}(hA^0 + \sum_{i=1}^m \sqrt{h}\xi^i A^i)^j \right| \\ &\leq K \sum_{j=1}^{r+s} \sum_{i=1}^m \frac{1}{j!} \sum_{\rho=0}^j h^{j-\rho+\frac{\rho}{2}} |\mathbf{E}(\xi^{i\rho} - \zeta_h^{i\rho})| + O(h^{\lceil \frac{r+s}{2} \rceil + 1}). \end{aligned}$$

According to the distribution function of the random variable ξ^i , we have

$$\begin{aligned} &|\mathbf{E}(X - \hat{X})| \\ &\leq K \sum_{j=1}^{r+s} \sum_{i=1}^m \frac{1}{j!} \sum_{\rho=\text{even}}^j h^{j-\rho+\frac{\rho}{2}} \left| \int_{A_h}^{+\infty} (x^\rho - A_h^\rho) \exp(-\frac{x^2}{2}) dx \right| + O(h^{\lceil \frac{r+s}{2} \rceil + 1}) \\ &\leq K \sum_{j=1}^{r+s} \frac{1}{j!} \sum_{\rho=\text{even}, \geq 2}^j h^{j-\rho+\frac{\rho}{2}} \sum_{u=1}^{\rho-1} A_h^u \exp(-\frac{A_h^2}{2}) \int_0^{+\infty} x^{\rho-u} \exp(-\frac{x^2}{2}) dx + O(h^{\lceil \frac{r+s}{2} \rceil + 1}). \end{aligned}$$

From the condition $A_h^2 \geq 2\ell |\ln h|$ which implies $\exp(-\frac{A_h^2}{2}) \leq h^\ell$, we can get

$$|\mathbf{E}(X - \hat{X})| \leq K \sum_{j=1}^{r+s} \frac{1}{j!} \sum_{\rho=\text{even}}^j h^{j-\rho+\frac{1}{2}+\ell} + O(h^{\lceil \frac{r+s}{2} \rceil + 1}) = O(h^{\lceil \frac{r+s}{2} \rceil + 1}).$$

Secondly, we estimate $\mathbf{E}|(X - \hat{X})|^2$,

$$\begin{aligned} & \mathbf{E}|(X - \hat{X})|^2 \\ & \leq K \mathbf{E} \left| \sum_{j=1}^{r+s} \frac{1}{j!} [(hA^0 + \sum_{i=1}^m \sqrt{h} \xi^i A^i)^j - (hA^0 + \sum_{i=1}^m \sqrt{h} \zeta_h^i A^i)^j] \right|^2 \\ & + K \mathbf{E} \left| \sum_{j=r+s+1}^{+\infty} \frac{1}{j!} (hA^0 + \sum_{i=1}^m \sqrt{h} \xi^i A^i)^j \right|^2 \\ & \leq K \sum_{j=1}^{r+s} \sum_{i=1}^m \frac{1}{j!} \sum_{\rho=0}^j h^{2j-\rho} \mathbf{E}|(\xi^{i\rho} - \zeta_h^{i\rho})|^2 + O(h^{r+s+1}) \\ & \leq K \sum_{j=1}^{r+s} \sum_{i=1}^m \frac{1}{j!} \sum_{\rho=\text{even}}^j h^{2j-\rho} \left| \int_{A_h}^{+\infty} (x^\rho - A_h^\rho)^2 \exp(-\frac{x^2}{2}) dx \right| + O(h^{r+s+1}) \quad (27) \\ & \leq K \sum_{j=1}^{r+s} \frac{1}{j!} \sum_{\rho=\text{even}, \geq 0}^j h^{2j-\rho} \sum_{u=1}^{\rho-1} A_h^{2u} \exp(-\frac{A_h^2}{2}) \int_0^{+\infty} x^{2\rho-2u} \exp(-\frac{x^2}{2}) dx \\ & + O(h^{r+s+1}) \\ & \leq K \sum_{j=1}^{r+s} \frac{1}{j!} \sum_{\rho=\text{even}}^j h^{2j-2\rho+1+\ell} + O(h^{r+s+1}) \\ & = O(h^{r+s+1}), \end{aligned}$$

Thus, the numerical scheme (26) is of the mean-square order $\frac{r+s}{2}$. Recall the one-step approximation $X^{(r,s)}(t+h; t, x)$ (14) we proposed

$$X^{(r,s)} = \left[I + \sum_{j=1}^s b_j (-\bar{B})^j \right]^{-1} \left[I + \sum_{j=1}^r a_j \bar{B}^j \right] x, \quad (28)$$

where $\bar{B} = hA^0 + \sum_{i=1}^m \sqrt{h} \zeta_h^i A^i$. From (12), we know that

$$\begin{aligned} \hat{X} - X^{(r,s)} &= x + \sum_{j=1}^{r+s} \frac{1}{j!} \bar{B}^j x - P_{(r,s)}(\bar{B})x \\ &= \exp(\bar{B})x - P_{(r,s)}(\bar{B})x - \sum_{j=r+s+1}^{+\infty} \frac{1}{j!} \bar{B}^j x \\ &= \sum_{j=r+s+1}^{+\infty} \bar{c}_j \bar{B}^j x, \end{aligned} \quad (29)$$

where $\bar{c}_j, j \geq r + s + 1$, are constants. It is obvious that

$$\begin{aligned} |\mathbf{E}(\hat{X} - X^{(r,s)})| &\leq K_1 |\mathbf{E} \sum_{j=r+s+1}^{+\infty} (hA^0 + \sum_{i=1}^m \sqrt{h} \zeta_h^i A^i)^j| \\ &= O(h^{\lfloor \frac{r+s}{2} \rfloor + 1}), \end{aligned} \quad (30)$$

and

$$\begin{aligned} \mathbf{E}|\hat{X} - \bar{X}^{(r,s)}|^2 &\leq K_1 \mathbf{E} \left| \sum_{j=r+s+1}^{+\infty} (hA^0 + \sum_{i=1}^m \sqrt{h} \zeta_h^i A^i)^j \right|^2 \\ &= O(h^{r+s+1}), \end{aligned} \quad (31)$$

where K_1 is a sufficiently large constant. Applying Proposition 4.2, we prove the theorem. \square

Theorem 4.2. The numerical method (21) based on the Padé approximation $P_{(\hat{r}, \hat{s})}$ and $P_{(\check{r}, \check{s})}$ as follows

$$\begin{aligned} Z_{n+1} &= \left[I + \sum_{j=1}^{\hat{s}} b_j (-B_1)^j \right]^{-1} \left[I + \sum_{j=1}^{\hat{r}} a_j (B_1)^j \right] Z_n \\ &\quad + \sum_{i=1}^m \int_{t_n}^{t_{n+1}} \left[I + \sum_{j=1}^{\check{s}} b_j (-B_2)^j \right]^{-1} \left[I + \sum_{j=1}^{\check{r}} a_j (B_2)^j \right] J^{-1} R_i dW^i(\theta), \end{aligned}$$

where $B_1 = hJ^{-1}\tilde{C}^0$ and $B_2 = (t_{n+1} - \theta)J^{-1}\tilde{C}^0$ with $\hat{r} + \hat{s} = \check{r} + \check{s} + 2$ and $\check{r}, \check{s} \geq 1$ is of mean-square order $\check{r} + \check{s} + 1$.

Proof. It is known that the solution $Z(t + h; t, z)$ of (19) is

$$Z = e^{hJ^{-1}\tilde{C}^0} z + \sum_{i=1}^m \int_t^{t+h} e^{(t+h-\theta)J^{-1}\tilde{C}^0} J^{-1} R_i dW^i(\theta), \quad (32)$$

and the one-step approximation $\bar{Z}(t + h; t, z)$ (21) is as follows

$$\bar{Z} = P_{\hat{r}, \hat{s}}(hJ^{-1}\tilde{C}^0)z + \sum_{i=1}^m \int_t^{t+h} P_{\check{r}, \check{s}}((t+h-\theta)J^{-1}\tilde{C}^0) J^{-1} R_i dW^i(\theta). \quad (33)$$

By estimating $|\mathbf{E}(Z - \bar{Z})|$ and $\mathbf{E}|(Z - \bar{Z})|^2$, we know that

$$|\mathbf{E}(Z - \bar{Z})| \leq K |e^{hJ^{-1}\tilde{C}^0} - P_{\hat{r}, \hat{s}}(hJ^{-1}\tilde{C}^0)| = O(h^{\hat{r} + \hat{s} + 1}). \quad (34)$$

and

$$\begin{aligned} &\mathbf{E}|(Z - \bar{Z})|^2 \\ &\leq K \mathbf{E} |e^{hJ^{-1}\tilde{C}^0} - P_{\hat{r}, \hat{s}}(hJ^{-1}\tilde{C}^0)|^2 \\ &\quad + K \mathbf{E} \left| \sum_{i=1}^m \int_t^{t+h} [e^{(t+h-\theta)J^{-1}\tilde{C}^0} - P_{\check{r}, \check{s}}((t+h-\theta)J^{-1}\tilde{C}^0)] J^{-1} R_i dW^i(\theta) \right|^2. \end{aligned} \quad (35)$$

Due to the independence of the Winner processes and the isometry's property of the Itô integral, the equation (35) becomes

$$\begin{aligned}
& \mathbf{E}|(Z - \bar{Z})|^2 \\
& \leq K \sum_{i=1}^m \int_t^{t+h} \mathbf{E} |[e^{(t+h-\theta)J^{-1}\tilde{C}^0} - P_{\tilde{r},\tilde{s}}((t+h-\theta)J^{-1}\tilde{C}^0)]J^{-1}R_i|^2 d\theta \\
& \quad + O(h^{2\tilde{r}+2\tilde{s}+2}) \\
& \leq K \sum_{i=1}^m \int_t^{t+h} \mathbf{E} |(t+h-\theta)^{\tilde{r}+\tilde{s}+1}|^2 d\theta + O(h^{2\tilde{r}+2\tilde{s}+5}) + O(h^{2\tilde{r}+2\tilde{s}+2}) \\
& = O(h^{2\tilde{r}+2\tilde{s}+3})
\end{aligned} \tag{36}$$

Thus, according to Proposition 4.1, the numerical scheme (21) is of the mean-square order $\tilde{r} + \tilde{s} + 1$. \square

Now let us prove the symplecticity of the schemes (15)~(18) and (21).

Theorem 4.3. The numerical methods $X_n^{(k,k)}$ e.g. (15)~(18) based on the Padé approximation are symplectic.

Proof. The proof of this theorem is a straight-forward extension of its counterpart in deterministic case in ([1]), and there is no essential difficulty arising from involving stochastic elements.

For the sake of simplicity, we also denote $\bar{B} = hA^0 + \sum_{i=1}^m \sqrt{h}\zeta_h^i A^i$ as above and consider the one-step approximation (14) based on $P_{(k,k)}(\bar{B})$ with $A_h^i = \sqrt{4k|\ln h|}$ as follows,

$$X_{n+1}^{(k,k)} = [D_{(k,k)}(\bar{B})]^{-1} [N_{(k,k)}(\bar{B})] X_n^{(k,k)}. \tag{37}$$

where

$$D_{(k,k)}(\bar{B}) = I + \sum_{j=1}^k a_j (-hA^0 - \sum_{i=1}^m \sqrt{h}\zeta_h^i A^i)^j,$$

and

$$N_{(k,k)}(\bar{B}) = I + \sum_{j=1}^k a_j (hA^0 + \sum_{i=1}^m \sqrt{h}\zeta_h^i A^i)^j.$$

Let $N_{(k,k)}(\bar{B}) = F(\bar{B}) + G(\bar{B})$, $D_{(k,k)}(\bar{B}) = F(\bar{B}) - G(\bar{B})$, where $F(\bar{B})$ is an even polynomial and $G(\bar{B})$ is an odd polynomial. Since $\bar{B} \in sp(2n)$, we get $F(\bar{B}^\top)J = JF(\bar{B})$ and $G(\bar{B}^\top)J + JG(\bar{B}) = 0$ according to the Theorem 2.1 and 2.2. We want to verify the numerical method $X_{n+1}^{(k,k)}$ is symplectic, so we should prove that the

matrix $D_{(k,k)}^{-1}N_{(k,k)} \in Sp(2n)$. Since

$$\begin{aligned}
& N_{(k,k)}^\top J N_{(k,k)} \\
&= (F(\bar{B}^\top) + G(\bar{B}^\top))J(F(\bar{B}) + G(\bar{B})) \\
&= J(F(\bar{B}) - G(\bar{B}))(F(\bar{B}) + G(\bar{B})) \\
&= J(F(\bar{B}) + G(\bar{B}))(F(\bar{B}) - G(\bar{B})) \\
&= (F(\bar{B}^\top) - G(\bar{B}^\top))J(F(\bar{B}) - G(\bar{B})) \\
&= D_{(k,k)}^\top J D_{(k,k)},
\end{aligned} \tag{38}$$

From the Theorem 2.3 we know that $D_{(k,k)}^{-1}N_{(k,k)} \in Sp(2n)$. Thus we prove the theorem. \square

Using arguments similar to ones in the proof of Theorem 4.3, we obtain the following theorem.

Theorem 4.4. If \hat{r} equals \hat{s} , the numerical methods Z_{n+1} (39) as follows

$$Z_{n+1} = P_{\hat{r},\hat{r}}(hJ^{-1}\tilde{C}^0)Z_n + \sum_{i=1}^m \int_{t_n}^{t_n+h} P_{\hat{r},\hat{s}}((t_n+h-\theta)J^{-1}\tilde{C}^0)J^{-1}R_i dW^i(\theta), \tag{39}$$

is symplectic.

Proof. It is known that the one-step approximation (39) is symplectic iff

$$\frac{\partial Z_{n+1}}{\partial Z_n}^\top J \frac{\partial Z_{n+1}}{\partial Z_n} = J.$$

Since $\frac{\partial Z_{n+1}}{\partial Z_n} = P_{\hat{r},\hat{r}}(hJ^{-1}\tilde{C}^0)$, the numerical method (39) is symplectic iff $P_{\hat{r},\hat{r}} \in Sp(2n)$. Repeating the proof as that of the Theorem 4.3, we prove the Theorem. \square

5 Numerical tests

Example 1 We consider the system of SDEs in the sense of Stratonovich, i.e. the Kubo oscillator

$$\begin{aligned}
dP &= -aQdt - \sigma Q \circ dW(t), \quad P(0) = p, \\
dQ &= aPdt + \sigma P \circ dW(t), \quad Q(0) = q,
\end{aligned} \tag{40}$$

where a and σ are constants and $W(t)$ is a one-dimensional standard Wiener process. The exact solution of (40) is

$$\begin{aligned}
P(t) &= p \cos(at + \sigma W(t)) - q \sin(at + \sigma W(t)), \\
Q(t) &= p \sin(at + \sigma W(t)) + q \cos(at + \sigma W(t)).
\end{aligned} \tag{41}$$

The phase flow of this system preserves symplectic structure. Moreover, the quantity $H(p, q) = p^2 + q^2$, is conservative for this system, i.e.:

$$H(P(t), Q(t)) = H(p, q), \quad \text{for } t \geq 0.$$

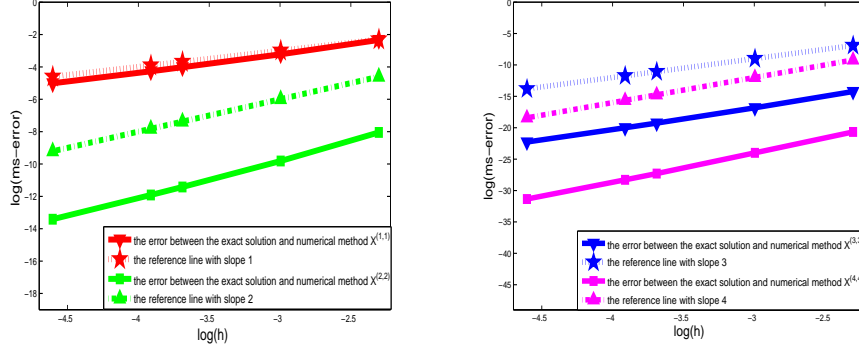


Figure 5.1: The mean-square convergence order of the scheme (42) and (43) (left), the mean-square convergence order of the scheme (44) and (45) (right).

This means that the phase trajectory of (40) is a circle centered at the origin.

We test here four methods. Denoting by $\bar{B}_i = (ah + \sigma\sqrt{h}\zeta_i)$, $i = 1, 2, 3, 4$, the symplectic methods (15)~(18) applied to (40) take the following forms, respectively,

$$\bar{X}_{n+1}^{(1,1)} = \bar{X}_n^{(1,1)} + \frac{1}{2}\bar{B}_1(\bar{X}_n^{(1,1)} + \bar{X}_{n+1}^{(1,1)}), \quad (42)$$

with $A_h^1 = \sqrt{4|\ln h|}$,

$$\bar{X}_{n+1}^{(2,2)} = \bar{X}_n^{(2,2)} + \frac{1}{2}\bar{B}_2(\bar{X}_n^{(2,2)} + \bar{X}_{n+1}^{(2,2)}) + \frac{1}{12}\bar{B}_2^2(\bar{X}_n^{(2,2)} - \bar{X}_{n+1}^{(2,2)}), \quad (43)$$

with $A_h^2 = \sqrt{8|\ln h|}$,

$$\bar{X}_{n+1}^{(3,3)} = \bar{X}_n^{(3,3)} + \left(\frac{1}{2}\bar{B}_3 + \frac{1}{120}\bar{B}_3^3\right)(\bar{X}_n^{(3,3)} + \bar{X}_{n+1}^{(3,3)}) + \frac{1}{10}\bar{B}_3^2(\bar{X}_n^{(3,3)} - \bar{X}_{n+1}^{(3,3)}), \quad (44)$$

with $A_h^3 = \sqrt{12|\ln h|}$, and

$$\bar{X}_{n+1}^{(4,4)} = \bar{X}_n^{(4,4)} + \left(\frac{1}{2}\bar{B}_4 + \frac{1}{84}\bar{B}_4^3\right)(\bar{X}_n^{(4,4)} + \bar{X}_{n+1}^{(4,4)}) + \left(\frac{1}{24}\bar{B}_4^2 + \frac{1}{1680}\bar{B}_4^4\right)(\bar{X}_n^{(4,4)} - \bar{X}_{n+1}^{(4,4)}). \quad (45)$$

with $A_h^4 = \sqrt{16|\ln h|}$.

The numerical tests examine the behaviors of the numerical methods from three aspects: first, the convergence rate of the numerical methods illustrated by Figure 1; second, the sample trajectory produced by the numerical methods and the true solution, as shown by Figure 2; and third, the Hamiltonians produced by the numerical methods, as presented in Figure 3.

Figure 5.1 shows that, comparing with the reference lines, the numerical method (42) is of mean-square order 1, the numerical schemes (43) and (44) are of the second and third mean square order respectively, and the numerical scheme (46) is of mean-square order 4. These validate the theorem regarding mean-square convergence order of the proposed methods. In our experiments we take $T = 5$, $p = 1$, $q = 0$

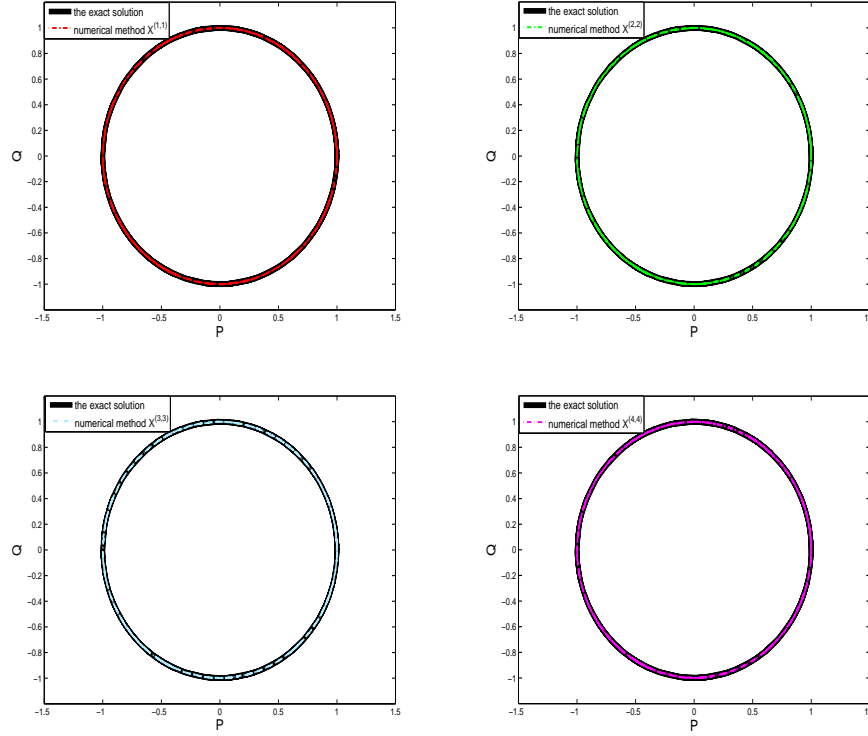


Figure 5.2: A sample path trajectory of the scheme (42) and (43) (above), A sample path trajectory of the scheme (44) and (45) (below).

and $h = [0.01, 0.02, 0.025, 0.05, 0.1]$. The expectation \mathbf{E} is approximated by taking average over 1000 sample paths.

Figure 5.2 gives approximations of a sample phase trajectory of (40) simulated by the symplectic methods (42)~(46). The initial condition is $p = 1, q = 0$. The corresponding exact phase trajectory belongs to the circle with the center at the origin and with the unit radius. We see that the symplectic methods are appropriate for simulation of the oscillator (40) on long time intervals $[0, 100]$.

It is not difficult to check that $H(P, Q)$ is conserved by the numerical methods (42)~(46). Figure 5.3 illustrate this fact as we can see the Hamiltonian of the numerical schemes (42)~(46) do not change. And we take $T = 100, p = 1, q = 0$ in our experiments.

Example 2 A linear stochastic oscillator.

The linear stochastic oscillator ([11])

$$\begin{aligned} dp &= -qdt + \sigma dW(t), & p(0) &= 0, \\ dq &= pdt, & q(0) &= 1 \end{aligned} \tag{46}$$

is a stochastic Hamiltonian system with $H = \frac{1}{2}(p^2 + q^2)$, $H_1 = -\sigma q$. Given the

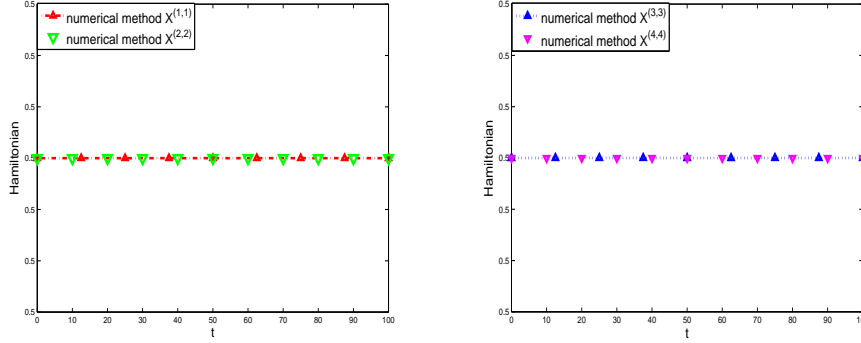


Figure 5.3: The conservation of the Hamiltonian of the scheme (42) and (43) (left), The conservation of the Hamiltonian of the scheme (44) and (45) (right)..

initial values, it can be shown ([11]) that the system (46) has the exact solution

$$\begin{aligned} p(t) &= p_0 \cos t + q_0 \sin t + \sigma \int_0^t \sin(t-s) dW(s), \\ q(t) &= -p_0 \sin t + q_0 \cos t + \sigma \int_0^t \cos(t-s) dW(s), \end{aligned} \quad (47)$$

which possess the two properties,

- (1) the second moment $\mathbf{E}(p(t)^2 + q(t)^2) = 1 + \sigma^2 t$;
- (2) (Markus and Weerasinghe [15]) $p(t)$ has infinitely many zeros, all simple, on each half line $[t_0, \infty)$ for every $t_0 \geq 0$, a.s..

Regarding the one-step approximations we proposed for the stochastic differential equations (46), if we replace the $P_{(\hat{r}, \hat{s})}$ and $P_{(\tilde{r}, \tilde{s})}$ in (21) with $P_{(2,2)}$ and $P_{(1,1)}$ respectively we will get

$$\begin{aligned} Z_{n+1} &= \left[I - \frac{h}{2} J + \frac{h^2}{12} J^2 \right]^{-1} \left[I + \frac{h}{2} J + \frac{h^2}{12} J^2 \right] Z_n \\ &\quad + \int_{t_n}^{t_{n+1}} \left[I - \frac{t_{n+1} - \theta}{2} J \right]^{-1} \left[I + \frac{t_{n+1} - \theta}{2} J \right] \begin{pmatrix} \sigma \\ 0 \end{pmatrix} dW(\theta), \end{aligned} \quad (48)$$

and if both $P_{(\hat{r}, \hat{s})}$ and $P_{(\tilde{r}, \tilde{s})}$ in (22) are taken place by $P_{(1,1)}$, the numerical methods as follows

$$Z_{n+1} = \left[I - \frac{h}{2} J \right]^{-1} \left[I + \frac{h}{2} J \right] Z_n + \left[I - \frac{h}{2} J \right]^{-1} \left[I + \frac{h}{2} J \right] \begin{pmatrix} \sigma \\ 0 \end{pmatrix} \Delta W_n \quad (49)$$

will be obtained.

With respect to the above numerical methods (48) and (49), our numerical tests focus on two aspects of view mainly. One is to sketch the numerical mean, i.e. the sample average of $p_n^2 + q_n^2$ directly, and then to compare their accordance with the reference line, the slop of which is the rate of the linear growth of the second moment of the solution (46). The other is to show that the proposed numerical schemes (48) and (49) preserve the oscillation property.

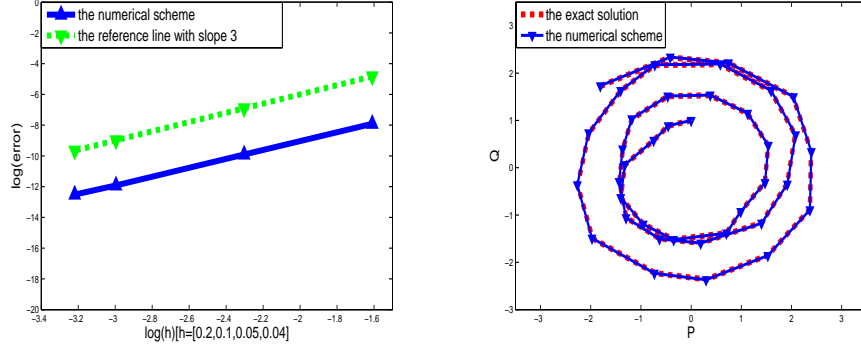


Figure 5.4: The mean-square convergence order of the scheme (48) (left), and a sample path trajectory of the numerical solution (48) (right).

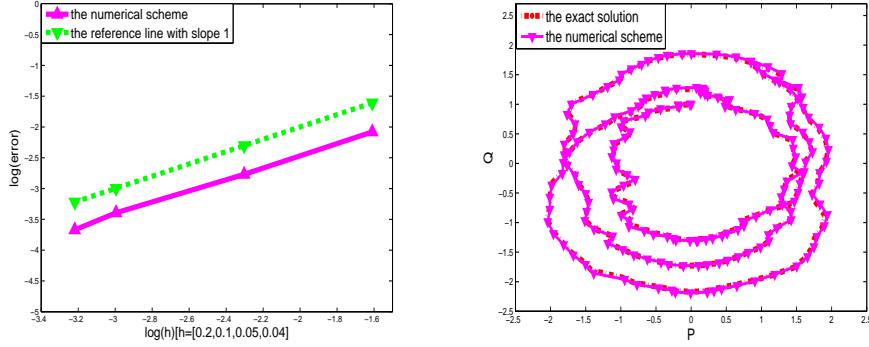


Figure 5.5: The mean-square convergence order of the scheme (49) (left), and a sample path trajectory of the numerical solution (49) (right).

For the numerical scheme (48), the left panel of Figure 5.4 plots the value $\ln |\mathbf{E}(P(T) + Q(T))^2 - \mathbf{E}(P_N + Q_N)^2|$ (blue-dotted line) against $\ln h$ for four different step sizes $h = [0.2; 0.1; 0.05; 0.04]$ at $T = 20$, where T is the subindex of one of the discrete time point such that $t_N = T$, and the $(P(T); Q(T))$ and $(P_N; Q_N)$ represent the phase point of the exact solution and the numerical method at time T , respectively. It can be seen that the mean-square order of the scheme (48) is 3, as indicated by the reference lines of slope 3. The expectation \mathbf{E} is approximated by taking average over 500 sample paths. The right panel draws one sample trajectory of the numerical method (48), and that of the stochastic differential equation (47), where near coincidence can be seen.

Similar to the test for the numerical scheme (48), we observe the mean-square convergence order of the numerical method (49) is 1, as illustrated by the left panel of Figure 5.5. The data setting for the left panel of Figure 5.5 is $p_0 = 0$, $q_0 = 1$, $T = 20$ and $h = [0.2; 0.1; 0.05; 0.04]$, and the expectation \mathbf{E} is approximated by taking average over 1000 sample paths. The right panel draws a sample phase trajectory of the exact solution (red-dotted line), the numerical method (49) (pink-solid line). The coincidence of a sample phase trajectory of the numerical method

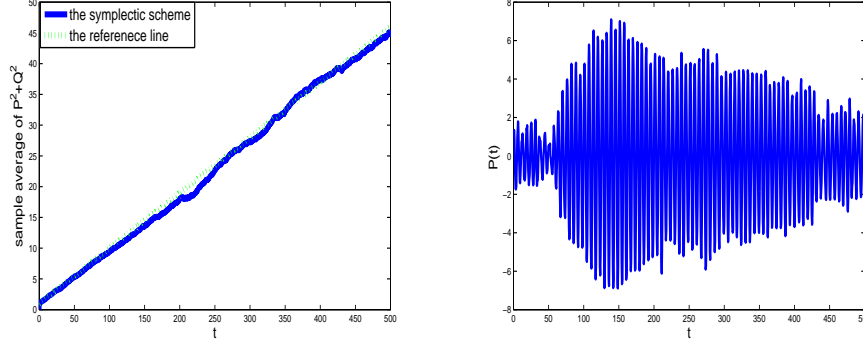


Figure 5.6: Preservation of the linear growth property of the numerical scheme (48) (left), Oscillation of the numerical solution (48) (right)..

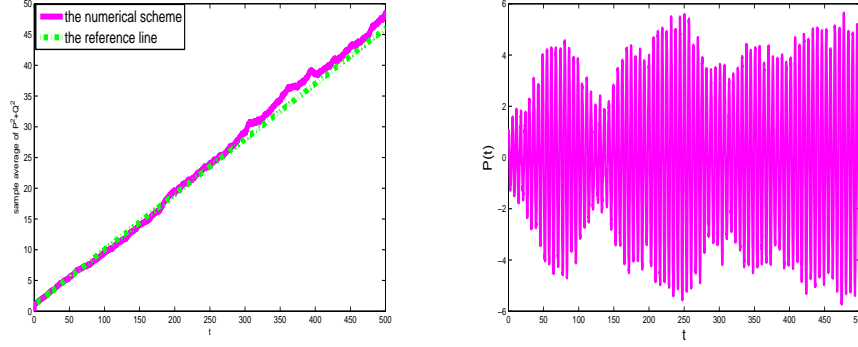


Figure 5.7: Preservation of the linear growth property of the numerical scheme (49) (left), Oscillation of the numerical solution (49) (right).

with that of the exact solution is also visible. The data here are the same with that for the right panel of Figure 5.4.

In the left panels of Figure 5.6 and 5.7, the quantity $\mathbf{E}(p_n^2 + q_n^2)$ with respect to the numerical solution $p_n, q_n, n = 0, 1, 2, \dots, 5000$ produced by the numerical schemes (48) (Figure 5.6) and (49) (Figure 5.7) is simulated through taking sample average over 500 numerical sample paths created by corresponding numerical schemes applied to the linear stochastic system 46. Here $\sigma = 0.3, t \in [0, 500]$ and the stepsize is 0.1. The reference straight line (green-dashed) has slope 0.09, which is equal to σ^2 . It can be seen from the two figures that both the numerical schemes (48) and (49) preserve the linear growth property of the second moment of the solution for the original system (46). Moreover, the long time oscillation behavior of the numerical solutions (48) and (49) are illustrated in the right panels of Figure 5.6 and 5.7. The property that the numerical solutions (48) and (49) have infinite zeros is obvious via the Figures.

6 Conclusion

We construct stochastic symplectic numerical methods using Padé approximation, for linear stochastic Hamiltonian systems, and special stochastic Hamiltonian systems with additive noises. Applications of the method to two examples, i.e Kubo oscillator and a linear stochastic oscillator, succeed in constructing symplectic numerical solutions based on Padé approximation which inherit the properties of the original systems. Numerical experiments show the mean-square convergence orders of the proposed schemes. For the deterministic situations, it is known that the numerical schemes based on the Padé approximation are A -stable ([10]) under appropriate conditions. However, stochastic stability of our methods still need further investigation.

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